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On the discrete version of Picone's identity

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Abstract

For a real number p with $1 < p$ we consider the first eigenvalues of the p-Laplacian on graphs, and estimates for the solutions of p-Laplace equations on graphs. We provide a discrete version of Picone's identity and its application. More precisely, we prove a Barta-type inequality for graphs with boundary. Finally, we provide a discrete version of the anti-maximum principle.

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1. Introduction

In the last decade there has been an increasing interest in the p-Laplacian, which plays an important role in geometry and partial differential equations. The p-Laplacian is a natural generalization of the Laplacian, which corresponds to $p = 2$. Although the Laplacian has been much studied, little is known about the nonlinear case $p \neq 2$. On the other hand, the discrete analogue of the Laplacian on Riemannian manifolds has recently been investigated. For the Laplacian on an infinite graph, Dodziuk and Kendall [8] proved a discrete analogue of a Cheeger-type inequality, which gave a lower bound for the bottom of the spectrum in terms of an isoperimetric constant.

The eigenvalues of the Laplacian have many physical interpretations, for example, as the frequencies of vibration of a membrane, as the rates of decay for the heat equation (or mass diffusion). In [14] are presented recently discovered connections between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of the underlying molecular graphs; other applications are given in [4–6]. So it is significant and necessary to investigate the relations between the graph-theoretic properties of a graph G and its eigenvalues.

However, the eigenvalues of the p-Laplacian can be calculated explicitly only for a few special graphs, most notably for regular graphs.

In this paper we study the discrete p-Laplacian, which is the analogue of the operator $\mathrm{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$. We will give some characterization and estimates of the first eigenvalue of the p-Laplacian and also generalize Picone's identity to the p-Laplacian for graphs. Some results concerning $-\Delta_p$ had long been known for the ordinary Laplacian, i.e. for $p = 2$: it is the purpose of this paper to continue in this direction but for the discrete p-Laplacian.

This paper is organized as follows. In Section 2, we define the discrete p-Laplacian. First eigenvalue for graphs with boundary is defined in Section 3. In Section 4 the minimum principle is given. Picone's identity for discrete graphs

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is advocated in Section 6. Section 4 deals with the minimum principle and Section 7 with discrete version of Barta's theorem. Finally, Section 9 concludes the paper and provides the anti-maximum principle for the discrete p-Laplacian.

2. Preliminaries

Let $G = (V, E)$ be a simple undirected graph with set of vertices V and set of unordered pairs of distinct elements of V called edges denoted by E. Two vertices x, y are adjacent if they are connected by an edge; in this case we write $x \sim y$.

A more familiar operator in the field of spectral geometry is the Laplacian operator, Δ, which appears so prominently in the heat equation and the wave equation. The analogue on G, $-\Delta$, can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \frac{1}{2}\sum_{x\sim y} |u(x) - u(y)|^2.$$

The factor $\frac{1}{2}$ appears since each edge is considered twice. In this way we define the Laplacian operator on G by

$$\Delta\phi(x) = \sum_{y\sim x}[\phi(x) - \phi(y)],$$

where ϕ is a function on the graph G. The basic idea of generalizing the discrete Laplacian is the following: instead of the uniform mass on Q, we assign to each edge $[x, y]$ a weight $c[x, y]$. We then obtain a new quadratic form Q_c defined by

$$Q_c(u) = \frac{1}{2}\sum_{x\sim y} c[x, y]|u(x) - u(y)|^2.$$

The analogue of Q_c in the continuous case on $\mathbb{R}^n$ is

$$Q_w(u) = \int_\Omega w(x)|\nabla u(x)|^2\,\mathrm{d}x,$$

where Ω is an open set of $\mathbb{R}^n$; and the quadratic differential operator induced by Q_w is denoted by $-\mathrm{div}(w\nabla u)$. The concept of the p-Laplacian Δ_p arises as a natural generalization of the Laplacian when we are interested in the discrete analogue case of the operator $-\mathrm{div}(w|\nabla\cdot|^{p-2}\nabla\cdot)$. Since it is natural to consider a Dirichlet form that decays with rate p (rather than the square), the relevant form is the *p-Dirichlet form* of a function u on G defined by

$$|u|_p^p = \frac{1}{2}\sum_{x\sim y} c[x, y]^{p-1}|u(x) - u(y)|^p.$$

In the same way we define the p-Laplacian of a function f at a vertex v by

$$\Delta_p f(v) = \sum_{u\sim v} c[x, y]^{p-1}(f(v) - f(u))^{p-1}.$$

In this paper we take $c[x, y] = 1$ for all $x \sim y$.

For an edge $\{u, v\}$ we choose an orientation $[u, v]$.

If f is a function on V, its *gradient* ∇f is the vector field E defined by the formula

$$\nabla f[u, v] = f(v) - f(u).$$

If U is a vector field on the oriented edges of E, its *divergence* denoted by $\mathrm{div}\, U$ is the function on V defined by the formula

$$\mathrm{div}\, U(v) = \sum_{u\sim v} U(v, u),$$

for a vertex v,

$$\Delta f(v) = \sum_{u \sim v} (f(v) - f(u)).$$

The Laplacian and the p-Laplacian of a function f on V $(1 < p < \infty)$ are the functions on V defined, respectively, as

$$\Delta f = -\mathrm{div}(\nabla f), \quad \varDelta_p f = -\mathrm{div}(|\nabla f|^{p-2} \nabla f).$$

Notation. In the following, for simplicity, we will use the expression

$$t^{\alpha} = t|t|^{\alpha-1}, \quad \alpha > 0, \;\; t \in \mathbb{R}.$$

We mean that $0^{\alpha} = 0$. Observe that, when $p - 1$ is an odd it coincides with usual notion.

Everywhere in the paper we shall use t^{p-1} for $t \in \mathbb{R}$ only with the meaning $t|t|^{p-2}$ and no other. Also we shall assume p and q are conjugate exponents; in other words, $(p-1)(q-1) = 1$.

With this notation the p-Laplacian of a function f at a vertex v is

$$\varDelta_p f(v) = \sum_{u \sim v} (f(v) - f(u))^{p-1}.$$

A real number λ is called an *eigenvalue* of $\varDelta_p$ if there exists a function $f \neq 0$ on V such that

$$\lambda f(v)^{p-1} = \sum_{u \sim v} (f(v) - f(u))^{p-1}$$

and the function f is called the corresponding *eigenfunction* to λ. Some results on the smallest positive eigenvalue are given in [2]. And this notion of eigenvalue generalizes the classical ($p = 2$ linear case) one which we found in [4]; other approaches can be found, for example, in [10,13,15].

Remark 2.1. Let $A(G)$ be the (0, 1)-adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. This coincides with our definition of the p-Laplacian when $p = 2$.

3. First eigenvalue for graphs with boundary

Let us first recall the discrete Laplacians. A graph with boundary means an undirected graph $G = (V \cup \partial V, E \cup \partial E)$ such that

(1) each edge in E has both endpoints in V,
(2) each edge in ∂E has exactly one endpoint in V and one in ∂V.

We call vertices in V (resp. ∂V) interior (resp. boundary) vertices, and similarly for the edges.

We fix a direction for each edge of G once and for all. Let $C_0(G)$ be the set of all real-valued functions f on $V \cup \partial V$ satisfying $f = 0$ on ∂V. Let $C^1(G)$ be the space of all functions ϕ on the set of directed edges of G satisfying $\phi([x, y]) = -\phi([y, x])$ where $[x, y]$, $x, y \in V \cup \partial V$, denotes a directed edge in $E \cup \partial E$ beginning at x and ending at y. We define $\|\nabla u\|_p^p$ and $\|u\|_p^p$ as follows:

$$\|\nabla u\|_p^p := |u|_p^p = \sum_{\{x,y\} \in E \cup \partial E} |u(x) - u(y)|^p$$

and

$$\|u\|_p^p := \sum_{x \in V \cup \partial V} |u(x)|^p.$$

A real number λ is an eigenvalue of Δ_p on $C_0(G)$ if there exists a function $0 \not\equiv f \in C_0(G)$ such that $\Delta_p f = \lambda f$ on V, where f is called the eigenfunction with the eigenvalue λ. It means that they satisfy the Dirichlet eigenvalue problem:

$$\begin{cases} \Delta_p f(x) = \lambda f(x)^{p-1}, & x \in V, \\ f(x) = 0, & x \in \partial V. \end{cases}$$

Remark 3.1. Let $G = (V, E)$ be a graph and let $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V \cup \{a\}$ and $\tilde{E} = E \cup \{\{v, a\}, v \in V\}$. Then λ is an eigenvalue of the graph G if and only if $(\lambda - 1)$ is an eigenvalue of the graph $\tilde{G}$ with boundary $\partial V = \{a\}$ and $\partial E = \{\{v, a\}, v \in V\}$.

As a consequence of this remark we get:

Proposition 3.2. *Let K_n be the complete graph. The eigenvalues of $\tilde{K}_n$ are*

$$(a^{q-1} + b^{q-1})^{p-1},$$

where $a, b \in \mathbb{N}^*$ such that $a + b \leqslant n$, and q the conjugate of p i.e. $(p-1)(q-1) = 1$.

Proof. This follows from the remark and the lemma of [2]. $\square$

We first consider the indefinite eigenvalue problem

$$\Delta_p u(x) = \lambda \mu(x) |u(x)|^{p-2} u(x),$$

where μ is a given weight positive function.

The following simple lemma will be also used in proving our main results.

Lemma 3.3. *In any graph with boundary, there is a first eigenfunction $u_\lambda \geqslant 0$ corresponding to the first eigenvalue λ.*

Proof. If u_0 minimizes

$$\inf \frac{\|\nabla u\|_p^p}{\sum \mu(x)|u(x)|^p},$$

where the inf is taken over all u with $u \not\equiv 0$ and $u = 0$ on ∂V, so does $|u_0|$. This shows the existence of a first eigenfunction of fixed sign. $\square$

Now we have enough machinery to state this section's main theorem.

Theorem 3.4. *The first eigenvalue is simple in any connected graph with boundary, i.e. all the associated first eigenfunctions are constant multiples of each other.*

The proof is in Section 6.

Remark 3.5. The result is not true for graphs without boundary. It suffices to take $p = 2$ and the complete graph K_n, Δ_2 has 0 as eigenvalue with multiplicity 1 and n as eigenvalue with multiplicity $n - 1$.

4. Minimum principle

Proposition 4.1. *Let u satisfy the differential inequality*

$$\Delta_p u(x) \geqslant 0$$

for all $x \in V$. If u attains a minimum at a point of V, then $u \equiv m = \inf u$ in V.

Proof. Assume that $m = u(v_0) = \inf_{v\in V} u(v)$ for some vertex v_0 of G, we have

$$\Delta_p u(v_0) = \sum_{v\sim v_0} (u(v_0) - u(v))^{p-1} \leqslant 0.$$

Combining this with $\Delta_p u(v_0) \geqslant 0$, we obtain $\Delta_p u(v_0) = 0$ and thus $u(v_0) = u(v)$ for all v such that $v \sim v_0$.

Let $X = \{v \in V,\ \text{such that } u(v) = m\}$. Suppose that $X \neq V$, we shall establish a contradiction. Let $v_t \in V\backslash X$. Since G is connected there exists a path $v_0, v_1, \ldots, v_n = v_t$ (i.e. sequence of vertices such that $v_i \sim v_{i+1}$ for $i = 0, 1, \ldots, n-1$). Let $k = \sup\{i, v_i \in X\}$. By applying the first part of the proof (replacing v_0 with v_k) we get $m = u(v_k) = u(v_{k+1})$. This contradicts the fact that $k = \sup\{i, v_i \in X\}$. Then $X = V$ and the proposition is proved. □

Remark 4.2. A maximum principle for functions $\Delta_p u \leqslant 0$ is obtained by applying the minimum principle to the function $-u$. Therefore, a nonconstant solution of the equation $\Delta_p u = 0$ can attain neither maximum nor a minimum at a vertex of V.

5. Rayleigh quotients

Suppose that $\sum_{x\in V} g(x) > 0$ and $\{x : g(x) < 0\} \neq \emptyset$. Consider the problem

$$\Delta_p u = \lambda g(x) u(x)^{p-1}.$$

The fact that $\sum_{x\in V} g(x) > 0$ implies

$$\lim_{c\longrightarrow\infty} \sum_{x\in V} g(x)|u(x) - c|^p = +\infty,$$

which shows that the function

$$c \mapsto \sum_{x\in V} g(x)|u(x) - c|^p$$

attains its minimum, i.e. there exists a $c_u \in \mathbb{R}$ such that

$$\sum_{x\in V} g(x)|u(x) - c_u|^p = \inf_c \sum_{x\in V} g(x)|u(x) - c|^p.$$

Define $\mathrm{Var}_{p,g}(u)$ by

$$\mathrm{Var}_{p,g}(u) := \inf_c \sum_{x\in V} g(x)|u(x) - c|^p.$$

The first positive eigenvalue λ_p^+ is given by

$$\lambda_p^+ = \inf \frac{|u|_p^p}{\mathrm{Var}_{p,g}(u)},$$

where the infimum is taken over all u with $\mathrm{Var}_{p,g}(u) > 0$.

The first negative eigenvalue λ_p^- is given by

$$\lambda_p^- = \sup \frac{|u|_p^p}{\mathrm{Var}_{p,g}(u)},$$

where the sup is taken over all u with $\mathrm{Var}_{p,g}(u) < 0$.

We note that the principal eigenvalues are characterized by the relation

$$\|\nabla\phi\|_p^p \geqslant \max(\lambda_p^- \mathrm{Var}_{p,g}(u), \lambda_p^+ \mathrm{Var}_{p,g}(u))$$

for all functions u. This is a Poincaré-type inequality.

Remark 5.1. When $\sum_{x\in V} g(x)<0$ and $\{x : g(x)>0\}\neq\emptyset$. We define $\mathrm{Var}_{p,g}(u)$ by $\mathrm{Var}_{p,g}(u) := \sup_c \sum_{x\in V} g(x) |u(x)-c|^p$.

Remark 5.2. In the case $\sum g(x)=0$, if we take, for example, a graph formed by two vertices a, b and edge connecting them, with $g(a)=-g(b)\neq 0$, then Var is indefinite and 0, the eigenvalue.

6. Picone's identity for graphs

The establishment of a Picone's identity has recently proved useful as a simple means of establishing a variety of results in spectral theory, Sturm comparison theorems, etc. for the p-Laplacian, $-\Delta_p$ and related equations. This version of the identity was exploited by Allegretto and Huang [1], where several other references are given. The aim of this section is to present a nonlinear discrete version of Picone's identity

$$|\nabla u|^2+\frac{u^2}{v^2}|\nabla v|^2-2\frac{u}{v}\nabla u\nabla v=|\nabla u|^2-\nabla\left(\frac{u^2}{v}\right)\nabla v\geqslant 0$$

for differentiable functions $v>0$ and $u\geqslant 0$. The discrete analogue of this identity is

$$|\nabla u(e)|^2-\nabla\left(\frac{u^2}{v}\right)(e)\nabla v(e)=v(x)v(y)\left|\nabla\frac{u}{v}(e)\right|^2.$$

We also present the discrete version for the p-Laplacian which states for $v>0$, $u\geqslant 0$ differentiable

$$R(u,v)=|\nabla u|^p-\nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v\geqslant 0.$$

Here, we are concerned with the discrete version of the p-Laplacian operator

$$-\mathrm{div}[|\nabla u|^{p-2}\nabla u]$$

with $p>1$. Note that for $p=2$ the p-Laplacian is the usual Laplace.

We formulate generalization of the Sturm–Picone theorem as follows:

Theorem 6.1. *Suppose u satisfies $\Delta_p u(x)\geqslant\lambda\mu(x)|u(x)|^{p-2}u(x)$ in V for some $\lambda>0$. Then for all w we have*

$$|w|_p^p\geqslant\lambda\sum_{x\in V} d(x)\mu(x)|w(x)|^p \tag{6.1}$$

and $\lambda\leqslant\lambda_p$. The equality in (6.1) holds if and only if $\lambda=\lambda_p$ and $w=ku$ for some constant k.

We need a lemma before proving Theorem 6.1:

Lemma 6.2. *For $v>0$ and $u\geqslant 0$, we have*

$$|\nabla u(e)|^p-\nabla\left(\frac{u^p}{v^{p-1}}(e)\right)|\nabla v(e)|^{p-2}\nabla v(e)\geqslant 0$$

with equality holding if and only if $\nabla(u/v)(e)=0$.

Proof. Let e be the oriented edge $[x,y]$. It suffices to establish that

$$|u(y)-u(x)|^p-\left(\frac{u(y)^p}{v(y)^{p-1}}-\frac{u(x)^p}{v(x)^{p-1}}\right)(v(y)-v(x))^{p-1}\geqslant 0.$$

If $v(y)>v(x)$, set $v(x)/v(y)=t$, then it suffices to show that

$$|u(y)-u(x)|^p\geqslant u(y)^p(1-t)^{p-1}-u(x)^p(1/t-1)^{p-1}.$$

This inequality is equivalent to

$$(1-t)\left(\frac{|u(y)-u(x)|^p}{(1-t)^p}\right)+t\frac{u(x)^p}{t^p}\geqslant u(y)^p$$

which follows from the convexity of the function $f(x)=x^p$. By symmetry we get the result when $v(x)>v(y)$, the case $v(x)=v(y)$, thus the first part of the lemma.

We remark that if $v(x)=v(y)$ then the equality holds if and only if $u(x)=u(y)$. Now suppose that $v(y)>v(x)$. By applying the same argument as the first part we get that the equality holds if and only is $u(x)/t=u(y)$ (since the function $f(x)=x^p$ is convex) and the result follows. $\square$

Proof of Theorem 6.1. For a function ψ on V with $\psi=0$ on ∂V, let $\phi=|\psi|$ and $\bar{E}=E\cup\partial E$, we have

$$\sum_{e\in\bar{E}}|\nabla\phi(e)|^p-\left[\nabla\frac{\phi^p}{u^{p-1}}(e)\right][\nabla u(e)]^{p-1}\geqslant 0$$

(we take $(\phi^p/u^{p-1})(x)=0$ for $x\in\partial V$). The second sum can be written as

$$-\sum_{x\sim y}\left[\frac{\phi^p}{u^{p-1}}(y)-\frac{\phi^p}{u^{p-1}}(x)\right][\nabla u(x,y)]^{p-1}.$$

By interchanging x and y in the first term, we see that this term is equal to the second one:

$$-2\sum\nolimits_{x\in V}\frac{\phi(x)^p}{u(x)^{p-1}}\varDelta_p u(x).$$

Therefore, the sum of these terms becomes

$$|\phi|_p^p-\sum_x\lambda\phi(x)^p\mu(x)\geqslant 0.$$

By the choice $\phi=u_1$ (with u_1 the first eigenvalue) we obtain $\lambda\leqslant\lambda_p$.

Suppose for some u_0 with $u_0=0$ on ∂V, we have

$$\|\nabla u_0\|_p^p=\sum_{x\in V}\lambda|u_0(x)|^p\mu(x).$$

Using Poincaré's inequality we get $\lambda\geqslant\lambda_p$, thus $\lambda=\lambda_p$, and by using the same arguments as in the first part we obtain

$$|\nabla u_0(e)|^p-\left[\nabla\frac{u_0^p}{u_1^{p-1}}(e)\right][\nabla u_1(e)]^{p-1}\geqslant 0.$$

By applying Lemma 6.2 we get $\nabla u_0/u_1=0$, and this completes the proof. $\square$

Proof of Theorem 3.4. Let u_1 be the first eigenfunction. By applying the lemma and minimum principle we can choose u_1 such that $u_1>0$ in V. If u is another eigenfunction, then we have $\|\nabla u\|_p^p=\lambda\|u\|_p^p$. By applying Theorem 6.1 we get the result. $\square$

7. Discrete version of Barta's theorem

Now consider the problem

$$\begin{cases}\varDelta_p u(x)=\lambda g(x)u(x)^{p-1}, & x\in V,\\ u(x)=0, & x\in\partial V.\end{cases}$$

If $g^+ \neq 0$ and $g^- \neq 0$ then there exist principal eigenvalues $\lambda_p^+ > 0 > \lambda_p^-$ such that the problem above has positive eigenfunctions associated with λ_p^+ and λ_p^-, respectively. We also have

$$\lambda_p^+ = \inf\left\{\frac{\|\nabla u\|_p^p}{\sum g(x)|u(x)|^p}, \sum g(x)|u(x)|^p > 0\right\}$$

and

$$\lambda_p^+ = \sup\left\{\frac{\|\nabla u\|_p^p}{\sum g(x)|u(x)|^p}, \sum g(x)|u(x)|^p < 0\right\}.$$

We have a Poincaré-type inequality which is

$$\|\nabla u\|_p^p \geqslant \max\left(\lambda_p^+ \sum_{x\in V} g(x)|u(x)|^p, \lambda_p^- \sum_{x\in V} g(x)|u(x)|^p\right)$$

for all functions u with $u = 0$ on ∂V.

Barta's theorem plays an essential role to obtain Cheng's comparison theorem; see, for example, [15]. In this section, we give its discrete version as follows:

Theorem 7.1. *Let $G = (V \cup \partial V, E \cup \partial E)$ be a finite connected graph with boundary. Let λ_1 be the first eigenvalue of the Dirichlet problem for Δ_p. Let f be a function on $V \cup \partial V$ such that f is positive on V and equal to zero on ∂V. Then*

$$\inf \frac{\Delta_p f}{f^{p-1}} \leqslant \lambda_1 \leqslant \sup \frac{\Delta_p f}{f^{p-1}}.$$

Proof. The proof in the discrete linear case proceeds in a similar way as in Barta's theorem for a Riemannian manifold; see [15]. For the nonlinear case, we give a proof here. Put $a = \sup\ \Delta_p f/f^{p-1}$, we have

$$\Delta_p f \leqslant a f^{p-1}.$$

Then

$$\sum_{x\in V} \Delta_p(x) f(x) \leqslant a \sum_{x\in V} f(x)^{p-1} f(x)$$

which gives

$$\|\nabla f\|_p^p \leqslant a\|f\|_p^p,$$

thus $\lambda_p \leqslant a$.

Let $m = \inf\ \Delta_p f/f^{p-1}$. If $m \leqslant 0$ the result is trivial. Suppose that $m > 0$. We have

$$\Delta_p f \geqslant m f^{p-1}.$$

By applying Theorem 6.1 we get $m \leqslant \lambda_p$ and thus the result follows. □

8. Application of Picone's theorem

From Gergorin's theorem (the linear case $p = 2$), it follows that its eigenvalues are nonnegative real numbers. Moreover, since its rows sum to 0, 0 is the smallest eigenvalue of $L(G)$. It is well known that if H is a subgraph of G, then $\lambda(H) \leqslant \lambda(G)$.

Let $\lambda_p(G)$ be the principal eigenvalue of (). We have the following theorem:

Theorem 8.1. *Suppose that $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ with $E_1 \subset E_2$. Let u_i be the eigenfunction associated with $\lambda_p(G_i)$. Then $\lambda_p(G_1) \leqslant \lambda_p(G_2)$. Moreover, if G_1 or G_2 is connected then the equality holds if and only if for all $e \in E_2 \backslash E_1$ we have $\nabla u_i(e) = 0$.*

Proof. Evidently for ϕ a function on V we have

$$\sum_{e\in E_1} |\nabla\phi(e)|^p \leqslant \sum_{e\in E_2} |\nabla\phi(e)|^p.$$

Then it follows that $\lambda_p(G_1)\leqslant\lambda_p(G_2)$. We then conclude that $u_1 = ku_2$ which is impossible since $E_1 \subset E_2$ and $E_1 \neq E_2$. □

Consider the equation

$$\Delta_p u - \lambda_p g(x)u(x)^{p-1} = h(x), \quad x \in V, \ u\geqslant 0, \ x \in \partial V, \tag{8.1}$$

where λ_p is the principal eigenvalue and $h(x)\geqslant 0$.

Theorem 8.2. *Eq.* (8.1) *has a solution if and only if* $h = 0$. *In this case* $u = ku_1$ *where* u_1 *is the first eigenvalue*.

Proof. The necessary part is trivial. Let u be a solution of Eq. (8.1), and put $\bar{E} := E \cup \partial E$. Note that by definition for $\phi \in C_0(G)$ we have

$$\frac{1}{2}\sum_{e\in\bar{E}}(\nabla u(e))^{p-1}\nabla\phi(e) - \lambda_p \sum_{x\in V} g(x)u(x)^{p-1}\phi(x) = \sum_{x\in V} h(x)\phi(x).$$

Choose $\phi = u^- = \max(-u, 0)$. Since $u\geqslant 0$ on ∂V we get $\phi = 0$ on ∂V. We have

$$\frac{1}{2}\sum_{e\in\bar{E}}(\nabla u(e))^{p-1}\nabla u^-(e) - \lambda_p \sum_{x\in V} g(x)u(x)^{p-1}u^-(x) = \sum_{x\in V} h(x)u^-(x).$$

We remark that $u^-(x)u(x)^{p-1} = -|u(x)|^p$ and for all real a, b we have

$$-(a-b)^{p-1}(a^- - b^-)\geqslant |a^- - b^-|^p$$

which gives

$$\begin{aligned} -\sum_{x\in V} h(x)u^-(x) = & -\frac{1}{2}\sum_{e\in\bar{E}}(\nabla u(e))^{p-1}\nabla u^-(e) - \lambda_p \sum_{x\in V} g(x)|u^-(x)|^p \\ & \geqslant \frac{1}{2}\sum_{e\in\bar{E}} |\nabla u^-(e)|^p - \lambda_p \sum_{x\in V} g(x)|u^-(x)|^p. \end{aligned}$$

This implies $u^-(x) = 0$ for all $x \in V$, or

$$\frac{1}{2}\sum_{e\in\bar{E}} |\nabla u^-(e)|^p - \lambda_p \sum_{x\in V} g(x)|u^-(x)|^p = 0$$

from which we conclude that u^- is a multiple of the principal eigenfunction. So $u^- > 0$ i.e. $u = -u^-$ and u is a multiple of principal eigenfunction.

Suppose that $u^- = 0$, i.e. $u = u^+\geqslant 0$. By the minimum principle Proposition 4.1 we have $u > 0$, and by using the same argument as in Theorem 6.1 we get

$$\|\nabla\phi\|_p^p - \lambda_p \sum g(x)\phi(x)^p - \sum \phi(x)^p \frac{h(x)}{u(x)^{p-1}} \geqslant 0.$$

By choosing $\phi = u_1$ where u_1 is the first eigenvalue we get

$$-\sum u_1(x)^p \frac{h(x)}{u(x)^{p-1}} \geqslant 0$$

which implies $h = 0$ in V, thus u is the first eigenvalue. This achieves the proof. □

9. Anti-maximum principle

Let us briefly recall the situation in the second order case with Dirichlet boundary conditions. Consider for $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary $\partial\Omega$ and $f \in L^p(\Omega)$, $p > 1$, the boundary value problem

$$\Delta u + \lambda u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega \tag{9.1}$$

and let λ_1 denote the first eigenvalue. Assume that the function *f* is nonnegative and positive on a set of positive measure. As is well known, the maximum principle implies that if $\lambda < \lambda_1$, then the solution u is positive. It is well known that if $p > n$ then there exists $\lambda_f > \lambda_1$ such that if $\lambda \in)\lambda_1, \lambda_f($, then the solution u of Eq. (9.1) is negative in Ω.

We first consider the Dirichlet eigenvalue problem

$$\varDelta_p u(x) = \lambda g(x)|u(x)|^{p-2}u(x), \quad x \in V \quad \text{and} \quad u = 0 \quad \text{on } \partial V, \tag{9.2}$$

where $g$is a given weight function.

We may formulate the result as a strengthened form of the proof linear case. Using a similar argument to that in the proof linear case we can prove the following:

Theorem 9.1. *Let λ_p be the principal eigenvalue of Eq.* (9.2). *Then for any $h \geqslant 0$, there exists $\varepsilon > 0$ such that if u_λ is a solution of*

$$\varDelta_p u - \lambda g(x) u(x)^{p-1} = h(x) \quad in\ V \quad u = 0 \quad on\ \partial V$$

with $\lambda \in]\lambda_p, \lambda_p + \varepsilon[$ then u_λ changes sign.

Proof. Let μ_h be defined by

$$v_h = \inf\{\|\nabla u\|_p^p + \sum h(x)|u(x)|,\ \operatorname{Var}_{p,g} u = 1,\ u = 0 \text{ on } \partial V\}.$$

Let $\lambda \in]\lambda_p, v_h[$. If u_λ is solution of Eq. (9.2), we have to show that $u_\lambda \geqslant 0$ does not hold everywhere in V. We will proceed by contradiction. Suppose $u_\lambda \geqslant 0$. Since $u_\lambda \geqslant 0$ and not identical to 0 it follows from the minimum principle (Proposition 4.1) that $u_\lambda > 0$.

If $u > 0$ we apply Theorem 6.1 and we obtain a contradiction. Then the theorem is proved. □

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